Problem Set # 1 Solutions

Problem 1 H&M # 5.6 charge conjugation operator

Representation (5.4) of γ-matrices : $\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$, $\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$
(Dirac-Pauli)

To show: C s.t. $C\gamma^0 = i\gamma^2 = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix}$

satisfies $-(C\gamma^0)\gamma^{\mu *} = \gamma^\mu (C\gamma^0)$

Just check explicitly in D-P rep , $\gamma^\mu \equiv (\beta, \beta\vec{\alpha})$

$$\gamma^0 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} & & & -i \\ & & i & \\ & i & & \\ -i & & & \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

γ^0 :

$$\text{LHS} = \begin{pmatrix} & & & -1 \\ & & 1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{RHS} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}\begin{pmatrix} & & & -1 \\ & & 1 & \\ & 1 & & \\ -1 & & & \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = \text{LHS} \checkmark$$

γ^1 :

$$\text{LHS} = \begin{pmatrix} & & & -1 \\ & & 1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}\begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\text{RHS} = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}\begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \text{LHS} \checkmark$$

γ^2:

$$\text{LHS} = \begin{pmatrix} & & & -1 \\ & & 1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}\begin{pmatrix} & & & i \\ & & -i & \\ & -i & & \\ i & & & \end{pmatrix} = \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}$$

$$\text{RHS} = \begin{pmatrix} & & & -i \\ & & i & \\ & i & & \\ -i & & & \end{pmatrix}\begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix} = \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} = \text{LHS} \checkmark$$

γ^3:

$$\text{LHS} = \begin{pmatrix} & & & -1 \\ & & 1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\text{RHS} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = \text{LHS} \checkmark$$

$$\Rightarrow \boxed{-(C\gamma^0)\gamma^{\mu *} = \gamma^\mu (C\gamma^0)} \quad \text{as required}$$

This can also be shown more concisely as follows:

For $\mu = 0$, $\gamma^{0*} = \gamma^0$

so $-(C\gamma^0)\gamma^{0*} = -i\gamma^2\gamma^0 = i\gamma^0\gamma^2 = \gamma^0(i\gamma^2) = \gamma^0(C\gamma^0)$

Sim for $\mu = 1, 3$

For $\mu = 2$, $\gamma^{2*} = -\gamma^2$

$-(C\gamma^0)\gamma^{2*} = i\gamma^2\gamma^2 = \gamma^2(i\gamma^2) = \gamma^2(C\gamma^0)$

So again, for all μ, $-(C\gamma^0)\gamma^{\mu *} = \gamma^\mu(C\gamma^0)$

Now apply C to a spinor.

To show: $\psi_c^{(1)} = i\gamma^2 \left[u^{(1)}(\vec{p})\, e^{-ip\cdot x}\right]^* = u^{(4)}(-\vec{p})\, e^{-ip\cdot x}$
$= v^{(1)}(\vec{p})\, e^{ip\cdot x}$

$$\text{LHS} = i\gamma^2 \left[u^{(1)}(\vec{p})\, e^{-ip\cdot x}\right]^* = i\gamma^2 \left[u^{(1)*}(p)\, e^{ip\cdot x}\right]$$

$\vec{\sigma}\cdot\vec{p} = \begin{bmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{bmatrix}$

$$= i\gamma^2 N \begin{bmatrix} \chi^{(1)*} \\ \left(\dfrac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi^{(1)}\right)^* \end{bmatrix} e^{ip\cdot x} \qquad \chi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= N \begin{bmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{bmatrix}^* e^{ip\cdot x} = Ni\gamma^2 \begin{bmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x-ip_y}{E+m} \end{bmatrix} e^{ip\cdot x}$$

$$\Rightarrow \quad \psi_c^{(1)} = N \begin{bmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{bmatrix} e^{ip\cdot x}$$

compare

$$u^{(4)}(\vec{p}) = N \begin{bmatrix} \frac{-\vec{\sigma}\cdot\vec{p}}{|E|+m}\chi^{(2)} \\ \chi^{(2)} \end{bmatrix} = N \begin{bmatrix} \frac{-p_x+ip_y}{|E|+m} \\ \frac{p_z}{|E|+m} \\ 0 \\ 1 \end{bmatrix}$$

$$u^{(4)}(-\vec{p}) = N \begin{bmatrix} \frac{p_x-ip_y}{|E|+m} \\ \frac{-p_z}{|E|+m} \\ 0 \\ 1 \end{bmatrix}$$

The antiparticle spinor is defined to be: (see 5.33)

$$v^{(1)}(\vec{p})\, e^{ip\cdot x} \equiv u^{(4)}(-\vec{p})\, e^{-i(-p)\cdot x}$$

So we have:

$$\boxed{\psi_c^{(1)} = u^{(4)}(-\vec{p})\, e^{ip\cdot x} = v^{(1)}(\vec{p})\, e^{ip\cdot x}}$$

as required

Next, show that
$$\begin{cases} C^{-1}\gamma^\mu C = (-\gamma^\mu)^T & \text{(A)} \\ C = -C^{-1} = -C^\dagger = -C^T & \text{(B)} \\ \bar{\psi}_c = -\psi^T C^{-1} & \text{(C)} \end{cases}$$

Many ways to show these...

(A) C is chosen s.t. $\quad -(C\gamma^0)\gamma^{\mu *} = \gamma^\mu (C\gamma^0)$

Multiply on right by $(C\gamma^0)^{-1}$

$$-(C\gamma^0)\gamma^{\mu *}(C\gamma^0)^{-1} = \gamma^\mu$$

$$-C\,\underbrace{\gamma^0\gamma^{\mu *}\gamma^0}\,C^{-1} = \gamma^\mu$$

$$\left.\begin{aligned} \gamma^{\mu\dagger} &= \gamma^0\gamma^\mu\gamma^0 \\ (\gamma^\mu)^{T*} &= \gamma^0\gamma^\mu\gamma^0 \\ (\gamma^\mu)^T &= \gamma^0\gamma^{\mu *}\gamma^0 \end{aligned}\right\}$$

$$-C\gamma^{\mu T}C^{-1} = \gamma^\mu$$

Multiply on RHS by C, on LHS by C^{-1}

$$\boxed{-(\gamma^\mu)^T = C^{-1}\gamma^\mu C} \quad \text{as req'd for (A)}$$

For $\mu = 0$, $\gamma^{0T} = \gamma^0$

$$(-\gamma^0)^T = C^{-1}\gamma^0 C \Rightarrow -\gamma^0 = C^{-1}\gamma^0 C$$

$$-C\gamma^0 = \gamma^0 C$$

From $(C\gamma^0)^T = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}^T = C\gamma^0$, $\quad -(C\gamma^0)^T = \gamma^0 C$

$$-\gamma^{0T} C^T = \gamma^0 C$$

$$-\gamma^0 C^T = \gamma^0 C$$

$$\Rightarrow \boxed{C = -C^T}$$

Similarly $\boxed{C = -C^\dagger}$

$$(C\gamma^0)(C\gamma^0) = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix}\begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = 1$$

$$C^{-1}\gamma^0 C = -\gamma^0$$

$$\gamma^0 = -C\gamma^0 C^{-1}$$

$$(C\gamma^0)\gamma^0 = -\big(\underbrace{(C\gamma^0)(C\gamma^0)}_{1} C^{-1}\big)$$

$\gamma^{0\,2} = 1$ $$C(1) = -C^{-1}$$

$$\Rightarrow \boxed{C = -C^{-1}}$$ as req'd for B

Finally, $$\bar{\psi}_c = \psi_c^\dagger \gamma^0 = (C\gamma^0\psi^*)^\dagger \gamma^0$$

$$= \psi^T (C\gamma^0)^\dagger \gamma^0$$

$$= \psi^T C \underbrace{\gamma^0\gamma^0}_{1}$$

$$\boxed{\bar{\psi}_c = -\psi^T C^{-1}}$$ as req'd for C

Problem 2

a) Maxwell's equations

$$\left\{ \begin{array}{ll} \nabla\cdot\vec{E} = \rho & \nabla\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0 \\ \nabla\cdot\vec{B} = 0 & \nabla\times\vec{B} - \frac{\partial\vec{E}}{\partial t} = \vec{J} \end{array} \right\}$$

The field strength tensor is

4-current

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}, \quad J^\mu = (\rho, \vec{J})$$

Consider $\partial_\mu F^{\mu\nu} = J^\nu$

$$\partial^0 F^{0\nu} + \partial^1 F^{1\nu} + \partial^2 F^{2\nu} + \partial^3 F^{3\nu} = J^\nu$$

$\nu = 0$ $\quad \frac{\partial}{\partial t}(0) + \frac{\partial}{\partial x}E_x + \frac{\partial}{\partial y}E_y + \frac{\partial}{\partial z}E_z = J^0 = \rho$

$$\boxed{\nabla\cdot\vec{E} = \rho} \qquad \text{Gauss' Law}$$

$\nu = 1$ $\quad \partial^0 F^{01} + \partial^1 F^{11} + \partial^2 F^{21} + \partial^3 F^{31} = J^1$

$$-\frac{\partial}{\partial t}E_x + 0 + \frac{\partial}{\partial y}B_z - \frac{\partial}{\partial z}B_y = J_x$$

$\nu = 2$ $\quad -\frac{\partial}{\partial t}E_y - \frac{\partial}{\partial x}B_z + 0 + \frac{\partial}{\partial z}B_x = J_y$

$\nu = 3$ $\quad -\frac{\partial}{\partial t}E_z + \frac{\partial}{\partial x}B_y - \frac{\partial}{\partial y}B_x + 0 = J_z$

$$\partial_\mu = \frac{\partial}{\partial x^\mu}$$
$$x^\mu = (t, x, y, z)$$
$$\partial^\mu = \frac{\partial}{\partial x_\mu}$$
$$x_\mu = (t, -x, -y, -z)$$
$$\partial_\mu a^\mu = \frac{\partial a_0}{\partial t} + \nabla\cdot\vec{a}$$
$$J^\mu = (\rho, J_x, J_y, J_z)$$

$$\nabla\times\vec{B} = \left(\frac{\partial}{\partial y}B_z - \frac{\partial}{\partial z}B_y\right)\hat{x} + \left(\frac{\partial}{\partial z}B_x - \frac{\partial}{\partial x}B_z\right)\hat{y}$$
$$+ \left(\frac{\partial}{\partial x}B_y - \frac{\partial}{\partial y}B_z\right)\hat{z}$$

So these 3 equations are equivalent to

$$\boxed{\nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{J}}$$ Ampère's Law

Note that these are only 2 of the 4 equations (the "inhomogeneous" ones). The other two (the "homogeneous" ones) follow directly from the statement that $\vec{B}$ can be written as

$$\vec{B} = \nabla \times \vec{A}$$

For $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, $A^\mu = (\phi, \vec{A})$

Examining our tensor:

$$\left.\begin{aligned} B_x &= F^{32} = \partial^3 A^2 - \partial^2 A^3 = -\left(\frac{\partial}{\partial z} A_y - \frac{\partial}{\partial y} A_z\right) \\ B_y &= F^{13} = \partial^1 A^3 - \partial^3 A^1 = -\left(\frac{\partial}{\partial x} A_z - \frac{\partial}{\partial z} A_x\right) \\ B_z &= F^{21} = \partial^2 A^1 - \partial^1 A^2 = -\left(\frac{\partial}{\partial y} A_x - \frac{\partial}{\partial x} A_y\right) \end{aligned}\right\} \text{equivalent to } \vec{B} = \nabla \times \vec{A}$$

We can then immediately see that

$$\nabla \cdot \vec{B} = \nabla \cdot (\nabla \times \vec{A}) = 0$$

$$\boxed{\nabla \cdot \vec{B} = 0}$$ "No magnetic monopoles"

Finally, again looking at our tensor, in terms of $\vec{E}$ field components:

$$\left.\begin{aligned} E_x &= F^{10} = \partial^1 A^0 - \partial^0 A^1 = -\frac{\partial}{\partial x}\phi - \frac{\partial A_x}{\partial t} \\ E_y &= F^{20} = \partial^2 A^0 - \partial^0 A^2 = -\frac{\partial}{\partial y}\phi - \frac{\partial A_y}{\partial t} \\ E_z &= F^{30} = \partial^3 A^0 - \partial^0 A^3 = -\frac{\partial}{\partial z}\phi - \frac{\partial A_z}{\partial t} \end{aligned}\right\} \text{equiv to } \vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}$$

So $$\nabla \times \vec{E} = \nabla \times \left[-\nabla\phi - \frac{\partial \vec{A}}{\partial t}\right] = -\underbrace{\cancel{\nabla \times [\nabla\phi]}}_{0} - \frac{\partial}{\partial t}\left[\nabla \times \vec{A}\right]$$

$$\Rightarrow \boxed{\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0}$$ Faraday's Law

b) $$\partial_\nu J^\nu = \partial_\nu \left[\partial_\mu F^{\mu\nu} \right]$$
$$= -\partial_\nu \left[\partial_\mu F^{\nu\mu} \right]$$

$F^{\mu\nu}$ is antisymmetric, i.e. $F^{\mu\nu} = -F^{\nu\mu}$

$$= - \partial_\mu \partial_\nu F^{\nu\mu}$$

$$\partial_\nu J^\nu = - \partial_\mu J^\mu$$

since this is just a summation over indices, $\partial_\mu J^\mu = \partial_\nu J^\nu$

$$\partial_\nu J^\nu = - \partial_\nu J^\nu$$

For this to be true, we must have $\boxed{\partial_\nu J^\nu = 0}$ equation of continuity

Problem 3

$$\mathcal{L} = i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi - (q\bar{\psi}\gamma^\mu\psi)A_\mu$$

"Coordinate" is $\bar{\psi}$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} = 0 \quad ; \quad \frac{\partial \mathcal{L}}{\partial(\bar{\psi})} = i\gamma^\mu \partial_\mu \psi - m\psi - q\gamma^\mu \psi A_\mu$$

Euler-Lagrange $\Rightarrow$ $\boxed{i\gamma^\mu \partial_\mu \psi - m\psi - q\gamma^\mu\psi A_\mu = 0}$

(Note this does not include a "free field" term for the vector field)

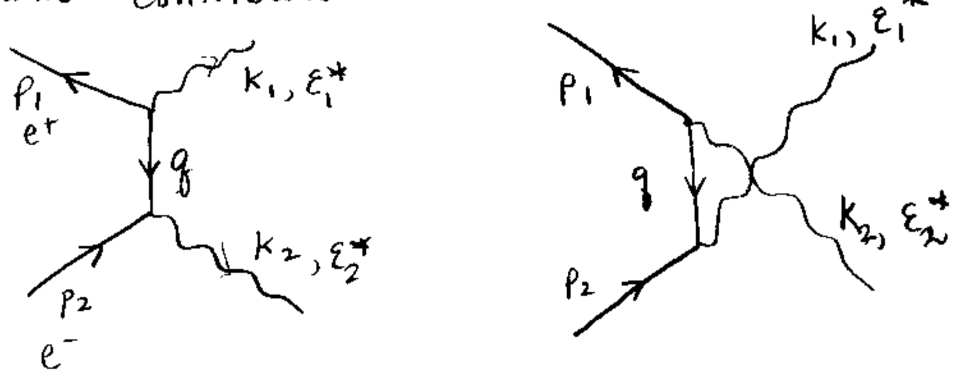
(Adjoint also acceptable as an answer)

Problem 4

$$e^+ + e^- \rightarrow \gamma + \gamma$$

Just follow F.R.'s

2 diagrams contribute:



First diagram : $u(p_2)$ for incoming electron

$\bar{v}(p_1)$ for incoming positron

$\varepsilon_1^*, \varepsilon_2^*$ for outgoing photons

Propagator: $\dfrac{i(\not{q} + m)}{q^2 - m^2}$

For each vertex: $ie\gamma^\mu$

4-momentum-conservation: $(2\pi)^4\,\delta^4(p_2 - k_2 + q)$ bottom vertex

$(2\pi)^4\,\delta^4(+p_1 - k_1 - q)$ top vertex

Sandwich everything together

in $\rightarrow$ +
out $\rightarrow$ −

$$(ie)^2\ \varepsilon^*_\mu(k_1)\gamma^\mu\bar{v}(p_1)\ \frac{i(\not{q}+m)}{q^2-m^2}\ \varepsilon^*_\nu(k_2)\gamma^\nu u(p_2)$$
$$\times (2\pi)^8\,\delta^4(p_2 - k_2 + q)\,\delta^4(q + p_1 - k_1)\,\frac{d^4 q}{(2\pi)^4}$$

Integrate over internal momenta: $q \rightarrow p_1 - k_1$

Get $-ie^2\ \varepsilon^*_\mu(k_1)\gamma^\mu\bar{v}(p_1)\left(\dfrac{\not{p}_1 - \not{k}_1 + m}{(p_1-k_1)^2 - m^2}\right)\varepsilon^*_\nu(k_2)\gamma^\nu u(p_2)\ \times$

$\times (2\pi)^4\,\delta^4(p_2 - k_2 + p_1 - k_1) \rightarrow$ get rid of this

Amplitude is

$$\boxed{-i\mathcal{M}_1 = -ie^2\ \varepsilon^*_\mu(k_1)\gamma^\mu\bar{v}(p_1)\left[\frac{\not{p}_1 - \not{k}_1 + m}{(p_1-k_1)^2 - m^2}\right]\varepsilon^*_\nu(k_2)\gamma^\nu u(p_2)}$$

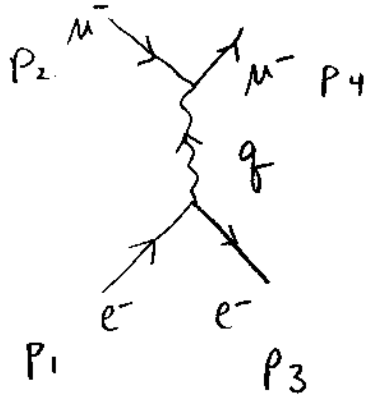
For the second diagram, just swap k_1 and k_2

$$\boxed{\mathcal{M}_2 = e^2 \, \varepsilon^*_\mu(k_2)\gamma^\mu \bar{v}(p_1)\left[\frac{\not p_1 - \not k_2 + m}{(p_1-k_2)^2 - m^2}\right]\varepsilon^*_\nu(k_1)\,\gamma^\nu u(p_2)}$$

$$\left.\boxed{\mathcal{M}_{tot} = \mathcal{M}_1 + \mathcal{M}_2}\right\}$$

<u>add</u> the two amplitudes (note <u>not</u> identical fermions in final state)

Problem 5   Electron - muon Scattering



Amplitude from F.R. is

$$\mathcal{M} = \frac{-e^2}{(p_1 - p_3)^2} \underbrace{\left[\bar{u}(p_3)\gamma^\mu u(p_1)\right]}_{\text{electron, mass } m} \underbrace{\left[\bar{u}(p_4)\gamma_\mu u(p_2)\right]}_{\text{muon, mass } M}$$

Applying Casimir's Trick

sum over spins

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\frac{e^4}{(p_1-p_3)^4} \mathrm{Tr}\left[\gamma^\mu(\not{p}_1 - m)\gamma^\nu(\not{p}_3 + m)\right] \times$$

$$\times \mathrm{Tr}\left[\gamma_\mu(\not{p}_2 + M)\gamma_\nu(\not{p}_4 + M)\right]$$

for average over initial spins

First trace: $\mathrm{Tr}\left[\gamma^\mu(\not{p}_1 - m)\gamma^\nu(\not{p}_3 + m)\right]$

$$= \underset{①}{\mathrm{Tr}(\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3)} + m\left[\underset{②}{\mathrm{Tr}(\gamma^\mu \not{p}_1 \gamma^\nu)} + \underset{③}{\mathrm{Tr}(\gamma^\mu\gamma^\nu\not{p}_3)}\right]$$

$$+ m^2 \underset{④}{\mathrm{Tr}(\gamma^\mu\gamma^\nu)}$$

Now deploy the trace theorems   ② & ③ are zero since they are the traces of odd no's of γ matrices.

$$④ \to Tr(\gamma^\mu \gamma^\nu) = 4 g^{\mu\nu}$$

$$① \to Tr(\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3) = (p_1)_\lambda (p_3)_\sigma \, Tr(\gamma^\mu \gamma^\lambda \gamma^\nu \gamma^\sigma)$$

using that $Tr(\gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma) = 4(g^{\mu\nu} g^{\lambda\sigma} - g^{\mu\lambda} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\lambda})$

$$① \to 4\left[(p_1)_\lambda (p_3)_\sigma \left(g^{\mu\lambda} g^{\nu\sigma} - g^{\mu\nu} g^{\lambda\sigma} - g^{\mu\sigma} g^{\lambda\nu}\right)\right]$$

$$= 4\left[p_1^{\mu} p_3^{\nu} - g^{\mu\nu} p_1 \cdot p_3 + p_3^{\mu} g_1^{\nu}\right]$$

And similarly for the second trace, $1 \to 2$; $3 \to 4$; $\mu \to \mu$; $m \to M$

Putting this all together, we get:

$$\langle |m|^2 \rangle = \frac{1}{4} \frac{e^4 \, 4 \cdot 4}{(p_1 - p_3)^4} \left[p_1^{\mu} p_3^{\nu} + p_3^{\mu} p_1^{\nu} + g^{\mu\nu}\left(m^2 - (p_1 \cdot p_3)\right)\right]$$

$$\times \left[p_{2\mu} p_{4\nu} + p_{4\mu} p_{2\nu} + g_{\mu\nu}\left(M^2 - (p_2 \cdot p_4)\right)\right]$$

In the limit that $m, M \to 0$

$$\langle |m|^2 \rangle = \frac{4 e^4}{(p_1 - p_3)^4} \Big[(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3)$$

$$- \cancel{(p_1 \cdot p_3)}(p_2 \cdot p_4) + (p_3 \cdot p_2)(p_1 \cdot p_4)$$

$$+ (p_3 \cdot p_4)(p_1 \cdot p_2) - \cancel{(p_3 \cdot p_1)}(p_2 \cdot p_4)$$

$$\cancel{-2(p_2 \cdot p_4)(p_1 \cdot p_3) + 4(p_1 \cdot p_3)}(p_2 \cdot p_4) \Big]$$

$$\langle |m|^2 \rangle = \frac{8e^4}{(p_1 - p_3)^4}\left[(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3)\right]$$

Now need to calculate the spin-averaged x-scn in the CM frame

$$\frac{d\sigma}{d\Omega} = \frac{1}{(8\pi)^2}\,\frac{\langle |m|^2 \rangle}{(E_e + E_\mu)^2}\,\frac{|\vec{p}_f|}{|\vec{p}_i|}$$

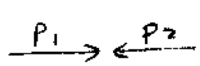
$|\vec{p}_f|$ is magnitude of either $|\vec{p}_3|$ or $|\vec{p}_4|$
$|\vec{p}_i|$ is magnitude of either $|\vec{p}_1|$ or $|\vec{p}_2|$

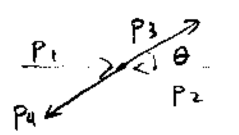
Assume everything nice & relativistic (neglect m & M, $E \sim |\vec{p}|$)

$p_1 = (E_e, \vec{p}_e)$ $\qquad$ $\vec{p}_3 = (E_e', \vec{p}_e')$
$p_2 = (E_\mu, \vec{p}_\mu)$ $\qquad$ $\vec{p}_4 = (E_\mu', \vec{p}_\mu')$

Rename:
$E = E_e$
$\vec{p} = \vec{p}_i = \vec{p}_e = -\vec{p}_\mu$
$\vec{p}\,' = \vec{p}_f = \vec{p}_{ef} = -\vec{p}_{\mu f}$

$p_1 = (E, \vec{p})$ $\qquad$ $p_3 = (E', \vec{p}\,')$
$p_2 = (E, -\vec{p})$ $\qquad$ $p_4 = (E', -\vec{p}\,')$





$$(p_1 - p_3)^2 = p_1^2 + p_3^2 - 2p_1 \cdot p_3$$
$$= E^2 - \vec{p}^{\,2} + E'^2 - \vec{p}\,'^2 - 2(EE' - \vec{p}_1 \cdot \vec{p}_3)$$

$$\vec{p} \cdot \vec{p}\,' = |\vec{p}||\vec{p}\,'| \cos\theta$$ scattering angle

$$(p_1 - p_3)^2 = -2E^2(1 - \cos\theta)^2$$

Conservation of 4-mom:

$$p_1 + p_2 = p_3 + p_4$$

$$E_1 + E_2 = E_3 + E_4$$

$$2E = 2E' \Rightarrow E = E'$$

And so $|\vec{p}\,'| = |\vec{p}|$ in relativistic limit

$$(p_1 \cdot p_2) = E^2 + |\vec{p}|^2 = 2E^2$$

$$(p_3 \cdot p_4) = 2E^2$$

$$(p_1 \cdot p_4) = E^2 - \vec{p}_1 \cdot \vec{p}_4 = E^2 + |\vec{p}|^2 \cos\theta = E^2(1+\cos\theta)$$

$$(p_2 \cdot p_3) = E^2 - \vec{p}_2 \cdot \vec{p}_3 = E^2 + |\vec{p}|^2 \cos\theta = E^2(1+\cos\theta)$$

Putting it together:

$$\frac{d\sigma}{d\Omega} = \frac{1}{(8\pi)^2} \frac{8e^4}{(2E)^2} \frac{\cancel{E}}{\cancel{E}} \left[\frac{4\cancel{E^4} + \cancel{E^4}(1+\cos\theta)^2}{(2\cancel{E^2}(1-\cos\theta))^2} \right]$$

$$= \frac{1}{(8\pi)^2} \frac{2e^4}{E^2} \left[\frac{4 + (1+\cos\theta)^2}{4(1-\cos\theta)^2} \right] = \frac{1}{(8\pi)^2} \frac{8e^4 \left[1 + \frac{1}{4}(1+\cos\theta)^2\right]}{4E^2(1-\cos\theta)^2}$$

Trig identities

$$(1-\cos\theta)^2 = 4\sin^4\frac{\theta}{2}$$

$$1 + \cos^4\theta/2 = 1 + \frac{1}{4}(1+\cos\theta)^2$$

$$\Rightarrow \quad \frac{d\sigma}{d\Omega} = \frac{1}{(8\pi)^2} \frac{e^4}{E^2} \frac{8\left[1 + \cos^4\theta/2\right]}{4\left(4 \sin^4\theta/2\right)}$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{(8\pi)^2} \frac{e^4}{2E^2} \frac{\left[1 + \cos^4\theta/2\right]}{\sin^4\theta/2}}$$