Problem Set 2 Solutions

Problem 1 Halzen & Martin 14.6

Show that $U(1)$ invariance of

$$\mathcal{L} = (\partial_\mu \phi)^* (\partial^\mu \phi) - m^2 \phi^* \phi$$

implies existence of a conserved current

$$J^\mu = -ie(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*) \qquad \text{(same as 3.25)}$$

$U(1)$ global invariance means

$$\phi \xrightarrow{U(1)} \phi' = e^{i\alpha} \phi$$

$$\phi^* \xrightarrow{U(1)} \phi^{*\prime} = e^{-i\alpha} \phi^*$$

$$\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} \delta(\partial^\mu\phi) + \frac{\partial \mathcal{L}}{\partial \phi^*} \delta\phi^* + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)^*} \delta((\partial_\mu \phi)^*)$$

$$\phi \to \phi' = \phi + i\alpha\phi + \cdots \qquad \delta\phi = i\alpha\phi$$

$$\phi \to \phi^{*\prime} = \phi^* - i\alpha\phi^* + \cdots \qquad \delta\phi^* = -i\alpha\phi^*$$

$$\partial_\mu \phi \to (\partial_\mu\phi)' = \partial_\mu\phi + i\alpha\partial_\mu\phi + \cdots \qquad \delta(\partial_\mu\phi) = i\alpha\partial_\mu\phi$$

$$\partial_\mu \phi^* \to (\partial_\mu\phi^*)' = \partial_\mu\phi^* - i\alpha(\partial_\mu\phi^*) \qquad \delta((\partial_\mu\phi)^*) = -i\alpha(\partial_\mu\phi)^*$$

From Euler-Lagrange

$$\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right] = \frac{\partial\mathcal{L}}{\partial\phi} \quad , \quad \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\right] = \frac{\partial\mathcal{L}}{\partial\phi^*}$$

Euler Lagrange equations

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\, i\alpha\phi + \frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}\, i\alpha\,\partial^\mu\phi + \frac{\partial\mathcal{L}}{\partial\phi^*}(-i\alpha\phi^*) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}(-i\alpha(\partial_\mu\phi)^*)$$

$$\partial^\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}\,\phi\right) = \partial^\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}\right)\phi + \frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}\,\partial^\mu\phi$$

$$\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}\,\partial^\mu\phi = \partial^\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}\,\phi\right) - \partial^\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}\right)\phi$$

Sim for ϕ^* ... plug in

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\, i\alpha\phi + i\alpha\left[-\partial^\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}\right)\phi + \partial^\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}\,\phi\right)\right]$$

$$-i\alpha\left[\frac{\partial\mathcal{L}}{\partial\phi^*}\,\phi^* - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\right)\phi^* + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi^*)}\,\phi^*\right)\right]$$

$$= i\alpha\underbrace{\cancel{\left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial^\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}\right)\right]}}_{\to\, 0 \text{ by E.L.}}\phi + i\alpha\,\partial^\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}\,\phi\right)$$

$$-i\alpha\underbrace{\cancel{\left[\frac{\partial\mathcal{L}}{\partial\phi^*} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\right)\right]}}_{\to\, 0 \text{ by E.L.}}\phi^* - i\alpha\,\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\,\phi^*\right)$$

Left with:

$$i\alpha\, \partial^\mu \underbrace{\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \phi - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \phi^* \right)} = 0$$

This corresponds to the conserved current j_μ, $\partial^\mu j_\mu = 0$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = (\partial^\mu \phi)^*$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} = \partial^\mu \phi$$

$\alpha \to e$ $$\boxed{j^\mu = ie\left[(\partial^\mu \phi)^* \phi - \phi^* \partial^\mu \phi\right]}$$ as required

Problem 2

Sorry about the algebra on this one. I did not take off any points for inaccuracies (hard to get it right...) The real point of this problem was to practice interpreting a Lagrangian.

$$\mathcal{L} = \underbrace{\frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) - \mu^2 \eta^2}_{\text{scalar mass } \sqrt{2}\mu} + \underbrace{\frac{1}{2}(\partial_\mu \xi)(\partial^\mu \xi)}_{\substack{\text{massless} \\ \text{Goldstone boson}}}$$

$$+ \underbrace{\frac{1}{2}\left(\frac{e\mu}{\lambda}\right)^2 A_\mu A^\mu - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}}_{\text{free } \underline{\text{massive}} \text{ gauge boson (Proca eqn)}}$$

$$- \left(\frac{e\mu}{\lambda}\right)(\partial_\mu \xi) A^\mu - e\left[\eta(\partial_\mu \xi) - \xi(\partial_\mu \eta)\right] A^\mu$$

no $2i$

$$+ \frac{\mu}{\lambda} e^2 \eta A_\mu A^\mu + \frac{1}{2} e^2 (\xi^2 + \eta^2) A_\mu A^\mu$$

$$\underbrace{- \lambda\mu(\eta^3 + \eta\xi^2) - \frac{1}{4}\lambda^2(\eta^4 + 2\eta^2\xi^2 + \xi^4)}_{\text{interactions}} + \underbrace{\left(\frac{\mu}{2\lambda}\right)^2}_{\text{constant, can ignore}}$$

The "Higgs mechanism" takes $\xi = 0$ $\Rightarrow$

So only ξ-free terms survive

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) - \mu^2 \eta^2 + \frac{1}{2}\left(\frac{e\mu}{\lambda}\right)^2 A_\mu A^\mu - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

$$+ \underbrace{\frac{\mu}{\lambda} e^2 \eta A_\mu A^\mu} + \underbrace{\frac{1}{2} e^2 \eta^2 A_\mu A^\mu} - \lambda\mu\eta^3 - \frac{1}{4}\lambda^2\eta^4$$

Surviving interactions

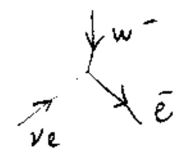
Problem 3 Halzen & Martin 12.2

Show that $\bar{u}_e \gamma^\mu \frac{1}{2}(1-\gamma^5) u_\nu$ (charge lowering weak current)

involves only LH e^-, RH e^+

In the limit s.t. $v \sim c$, show that electrons have $-$ helicity

$$\bar{u}_e \gamma^\mu \tfrac{1}{2}(1-\gamma^5) u_\nu = \bar{u}_e \tfrac{1}{2}(1+\gamma^5)\gamma^\mu u_\nu$$



LH electron $\quad u_{e_L} = \frac{1}{2}(1-\gamma^5) u_e$

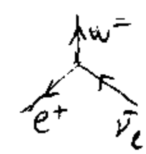
$$\bar{u}_{e_L} = \bar{u}_e \tfrac{1}{2}(1+\gamma^5)$$

$$\gamma_\mu\left(\frac{1-\gamma^5}{2}\right) = \left(\frac{1+\gamma^5}{2}\right)\gamma_\mu\left(\frac{1-\gamma^5}{2}\right)$$

$$J_W^- = \bar{u}_e \gamma^\mu \tfrac{1}{2}(1-\gamma^5) u_\nu = \bar{u}_e \tfrac{1}{2}(1+\gamma^5)\gamma^\mu \tfrac{1}{2}(1-\gamma^5) u_\nu$$

$$\Rightarrow \boxed{J_W^- = \bar{u}_{e_L} \gamma^\mu (u_\nu)_L}$$ involves only LH e^-

For positrons, $\quad v_{e^+ R} = \frac{1}{2}(1-\gamma^5) v_{e^+}$



$$J_W^- = \bar{v}_{\bar{\nu}} \tfrac{1}{2}\gamma^\mu (1-\gamma^5) v_{e^+}$$

$$= \bar{v}\, \tfrac{1}{2}(1+\gamma^5)\gamma^\mu \tfrac{1}{2}(1-\gamma^5) v_{e^+}$$

$$J_W^- = \bar{v}_{\bar{\nu}R} \gamma^\mu (v_{e^+})_R$$ couples RH $\bar{\nu}_e$ to RH e^+

4x4 matrix $\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$

Helicity operator: $\left(\frac{1}{2}\vec{\sigma}\cdot\hat{p}\right)$

$$u_L(p) = \frac{1}{2}(1-\gamma^5)u(p)$$

$$= \frac{1}{2}\left(1 - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)u(p)$$

electron $u(p) = \begin{pmatrix} u_A \\ u_B \end{pmatrix}$

where $u_A = \dfrac{(\vec{p}\cdot\vec{\sigma})}{E-m}u_B$

$u_B = \dfrac{(\vec{p}\cdot\vec{\sigma})}{E+m}u_A$

$$\gamma^5 u(p) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = \begin{pmatrix} u_B \\ u_A \end{pmatrix} = \begin{pmatrix} \frac{\vec{p}\cdot\vec{\sigma}}{E+m}u_A \\ u_A \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\vec{p}\cdot\vec{\sigma}}{E+m}u_A \\ \frac{\vec{p}\cdot\vec{\sigma}}{E-m}u_B \end{pmatrix}$$

$\bar{\nu}_e$

$$\gamma^5 u(p) = \begin{pmatrix} \frac{\vec{p}\cdot\vec{\sigma}}{E+m} & 0 \\ 0 & \frac{\vec{p}\cdot\vec{\sigma}}{E-m} \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

If $m=0$, $E=|\vec{p}|$

$$\gamma^5 u(p) = \begin{pmatrix} \hat{p}\cdot\vec{\sigma} & 0 \\ 0 & \hat{p}\cdot\vec{\sigma} \end{pmatrix}u(p) = \pm 1\, u(p)$$

$\rightarrow$ Helicity

$$u_L(p) = \frac{1}{2}(1-\gamma^5)u(p) = \frac{1}{2}(1-(\pm 1))u(p)$$

If helicity is positive, $u_L(p) = 0$

so $u_L(p)$ must correspond to a $\boxed{\text{negative helicity } e^-}$