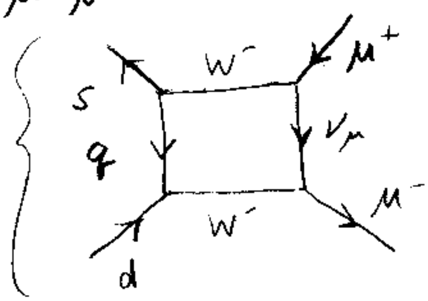
Problem Set #3 solutions

Problem 1

Show that the GIM mechanism works to suppress $K^0 \to \mu^+\mu^-$ for 3 quark generations, assuming that the mixing matrix U is unitary.

$K^0 \to \mu^+\mu^-$

K^0



For 3 generations

$q = u, c, t$

The matrix element for the LHS

$q = u \qquad m_1 = U_{ud} U_{us}^*$

$q = c \qquad m_2 = U_{cd} U_{cs}^*$

$q = t \qquad m_3 = U_{td} U_{ts}^*$

Tot: $m = U_{ud} U_{us}^* + U_{cd} U_{cs}^* + U_{td} U_{ts}^*$

If $U = \begin{pmatrix} U_{ud} & U_{us} & U_{ub} \\ U_{cd} & U_{cs} & U_{cb} \\ U_{td} & U_{ts} & U_{tb} \end{pmatrix}$ is unitary, $U^\dagger U = 1$

$$\begin{pmatrix} U_{ud}^* & U_{cd}^* & U_{td}^* \\ U_{us}^* & U_{cs}^* & U_{ts}^* \\ U_{ub}^* & U_{cb}^* & U_{tb}^* \end{pmatrix} \begin{pmatrix} U_{ud} & U_{us} & U_{ub} \\ U_{cd} & U_{cs} & U_{cb} \\ U_{td} & U_{ts} & U_{tb} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This requires $U_{us}^* U_{ud} + U_{cs}^* U_{cd} + U_{ts}^* U_{td} = 0$

Therefore if U is unitary

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 = 0$$

and because the amplitude vanishes (under the approximation of equal mass quarks), the decay is suppressed.

Problem 2 CP violation

Halzen & Martin 12.24

Verify 12.125) $\bar{u}_C = -u^T C^{-1}$

$$C^{-1}\gamma^\mu C = -(\gamma^\mu)^T$$

$$C^{-1}\gamma^\mu\gamma^5 C = +(\gamma^\mu\gamma^5)^T$$

using (5.39) $C^{-1}\gamma^\mu C = (-\gamma^\mu)^T$ (a)

$C = -C^{-1} = -C^\dagger = -C^T$ (b)

$\bar{\psi}_c = -\psi^T C^{-1}$ (c)

$\bar{u}_C = -u^T C^{-1}$ is equivalent to (5.39 c)

and $C^{-1}\gamma^\mu C = -(\gamma^\mu)^T$ is the same as (5.39 a)

Consider $\quad C^{-1}\gamma^\mu\gamma^5 C = +(\gamma^\mu\gamma^5)^T$

$$LHS = C^{-1}\gamma^\mu i\gamma^0\gamma^1\gamma^2\gamma^3 C$$

$$= C^{-1}\gamma^\mu(CC^{-1})i\gamma^0(CC^{-1})\gamma^1(CC^{-1})\gamma^2(CC^{-1})\gamma^3 C$$

Regroup

$$= (-\gamma^{\mu T})\, i\, (-\gamma^{0T})(-\gamma^{1T})(-\gamma^{2T})(-\gamma^{3T})$$

$$= -i\,\gamma^{\mu T}\gamma^{0T}\gamma^{1T}\gamma^{2T}\gamma^{3T}$$

$$= -\gamma^{\mu T}(i\gamma^3\gamma^2\gamma^1\gamma^0)^T$$

$$= -\gamma^{\mu T}(i\gamma^0\gamma^1\gamma^2\gamma^3)^T$$

$$= -\gamma^{\mu T}\gamma^{5T}$$

$$= -(\gamma^5\gamma^\mu)^T$$

$$= (\gamma^\mu\gamma^5)^T \quad = RHS$$

$$\Rightarrow \boxed{C^{-1}\gamma^\mu\gamma^5 C = (\gamma^\mu\gamma^5)^T}$$, as required

Problem 3   H & M section 13, 2

electroweak vertex for Z^0 coupling is

$$-i\left(g\cos\theta_W J_\mu^3 - g'\sin\theta_W \frac{j_\mu^Y}{2}\right)Z^\mu$$

$Z^0 \to f\bar{f}$ eqn 13.41 $\frac{-ig}{\cos\theta_W}\gamma^\mu \frac{1}{2}(c_V^f - c_A^f\gamma^5)$

Need to show: (Table 13.2)

	f	c_A^f	c_V^f
①	$\nu_e, \nu_\mu \ldots$	$\frac{1}{2}$	$\frac{1}{2}$
②	e^-, μ^-	$-\frac{1}{2}$	$-\frac{1}{2} + 2\sin^2\theta_W$
③	u, c	$\frac{1}{2}$	$\frac{1}{2} - \frac{4}{3}\sin^2\theta_W$
④	d, s	$-\frac{1}{2}$	$-\frac{1}{2} + \frac{2}{3}\sin^2\theta_W$

$g\sin\theta_W = g'\cos\theta_W$

$$-i\left(g\cos\theta_W J_\mu^3 - g'\sin\theta_W \frac{j_\mu^Y}{2}\right)Z^\mu$$

$$= -i\left(g\cos\theta_W J_\mu^3 - g\frac{\sin^2\theta_W}{\cos\theta_W}\frac{1}{2}(2j_\mu^{em} - 2J_\mu^3)\right)Z^\mu$$

$$= \frac{-ig}{\cos\theta_W}\left((\cos^2\theta_W + \sin^2\theta_W)J_\mu^3 - \sin^2\theta_W j_\mu^{em}\right)Z^\mu$$

$$= \frac{-ig}{\cos\theta_W}\left(J_\mu^3 - \sin^2\theta_W j_\mu^{em}\right)Z^\mu \qquad \text{H\&M (13.25)}$$

In terms of quantum numbers and non-chiral wavefunctions

$$(13.40) \quad \frac{-ig}{\cos\theta_W}\left(J^\mu - \sin^2\theta_W j_\mu^{em}\right) Z^\mu \qquad J_\mu^3 = \bar{\chi}_L \gamma_\mu \tfrac{1}{2}\tau_3 \chi_L$$

$$= -i\frac{g}{\cos\theta_W}\bar{\psi}_f\left[\gamma^\mu(1-\gamma^5)\frac{T}{2}^3 - \sin^2\theta_W Q\right]\psi_f Z_\mu$$

compare to

$$-i\frac{g}{\cos\theta_W}\frac{1}{2}\left(\gamma^\mu c_V^f - \gamma^\mu\gamma^5 c_A^f\right)$$

So
$$\boxed{\begin{aligned} c_V^f &= T^3 - 2\sin^2\theta_W Q \\ c_A^f &= T^3 \end{aligned}}$$

Neutrinos: $\left.\begin{aligned} T^3 &= \tfrac{1}{2} \\ Q &= 0 \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} c_A^f &= \tfrac{1}{2} \\ c_V^f &= \tfrac{1}{2} \end{aligned}\right.$

e^-, μ^-, τ^-: $\left.\begin{aligned} T^3 &= -\tfrac{1}{2} \\ Q &= -1 \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} c_A^f &= -\tfrac{1}{2} \\ c_V^f &= -\tfrac{1}{2} + 2\sin^2\theta_W \end{aligned}\right.$

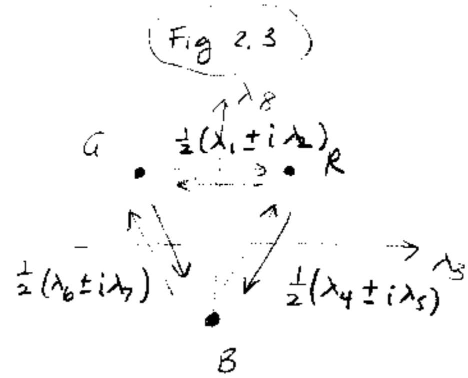
u, c, t: $\left.\begin{aligned} T^3 &= \tfrac{1}{2} \\ Q &= \tfrac{2}{3} \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} c_A^f &= \tfrac{1}{2} \\ c_V^f &= \tfrac{1}{2} - \tfrac{4}{3}\sin^2\theta_W \end{aligned}\right.$

d, s, b: $\left.\begin{aligned} T^3 &= -\tfrac{1}{2} \\ Q &= -\tfrac{1}{3} \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} c_A^f &= -\tfrac{1}{2} \\ c_V^f &= -\tfrac{1}{2} + \tfrac{2}{3}\sin^2\theta_W \end{aligned}\right.$

Problem 4

H&M 2.8

Gell-Mann matrices λ_i



'Obtain' is rather vague. you can write them down starting in many ways.

Starting from (2.43) $\lambda_3 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$, $\lambda_8 = \sqrt{\frac{1}{3}} \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$

Eigenvectors $R = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $G = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

The figure shows how R, G, B are related via 'raising' and 'lowering' operators built out of λ matrices

We want to get $R = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ from $G = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ via $\frac{1}{2}(\lambda_1 + i\lambda_2)$

This should look like the Pauli matrix operator which 'raises' $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

($\rightarrow$ but here, ignore the third component)

For Pauli matrices, it would be $\frac{1}{2}(\sigma_1 + i\sigma_2)$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

The corresponding λ matrices are then

(3rd row & column zero) $\boxed{\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}$

Similarly, we want to get between R & B $\begin{pmatrix}1\\0\\0\end{pmatrix}$ $\begin{pmatrix}0\\0\\1\end{pmatrix}$

with $\frac{1}{2}(\lambda_4 \pm i\lambda_5)$.

but ignore these

In this case, use the Pauli matrices, but with second row and column zero

$$\boxed{\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}}$$

And sim for λ_6, λ_7:

$$\boxed{\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}}$$

Next, show that $\left[\frac{\lambda_i}{2}, \frac{\lambda_j}{2}\right] = i \sum_k f_{ijk} \frac{\lambda_k}{2}$

where f_{ijk} are fully antisymmetric under interchange of any pair of indices;
non-vanishing values are permutations of

$$f_{123} = 1, \quad f_{458} = f_{678} = \frac{\sqrt{3}}{2}$$

$$\underbrace{f_{147} = f_{165} = f_{246} = f_{257} = f_{345} = f_{376} = \frac{1}{2}}$$

We'll just check explicitly only the cases corresponding to these

LHS $\left[\frac{\lambda_1}{2}, \frac{\lambda_2}{2}\right] = \frac{1}{2}\cdot\frac{1}{2}\left[\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix} - \begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}\right]$

$i=1, j=2$

$$= \frac{1}{4}\left[\begin{pmatrix}i&0&0\\0&-i&0\\0&0&0\end{pmatrix} - \begin{pmatrix}-i&0&0\\0&i&0\\0&0&0\end{pmatrix}\right] = \frac{1}{2}i\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}$$

$$= i f \frac{\lambda_3}{2}$$

This equals RHS for $\boxed{f = f_{123} = 1}$

$k = 4,5,6,7,8 \Rightarrow f_{124} = f_{125} = f_{126} = f_{127} = f_{128} = 0$

and clearly $j=1, i=2 \Rightarrow f_{213} = -1$

(antisymmetry under exchange of any 2 structure function indices follows similarly in all subsequent cases)

($i=4, j=5$)

$$\left[\frac{\lambda_4}{2}, \frac{\lambda_5}{2}\right] = \frac{1}{4}\left[\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}\begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix} - \begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix}\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}\right]$$

$$= \frac{1}{4}\left[\begin{pmatrix}i&0&0\\0&0&0\\0&0&-i\end{pmatrix} - \begin{pmatrix}-i&0&0\\0&0&0\\0&0&i\end{pmatrix}\right] = \frac{1}{2}\begin{bmatrix}i&0&0\\0&0&0\\0&0&-i\end{bmatrix}$$

$$= \frac{i}{2}\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix} = \frac{i}{2}\left\{ f \underbrace{\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}}_{\lambda_3} + f' \underbrace{\frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}}_{\lambda_8} \right\}$$

equals RHS for $f = \boxed{f_{453} = \frac{1}{2}}$, $f' = \boxed{f_{458} = \frac{\sqrt{3}}{2}}$

$f_{451} = f_{452} = f_{454} = f_{456} = f_{457} = 0$

$i=6,\ j=7$

$$\left[\frac{\lambda_6}{2}, \frac{\lambda_7}{2}\right] = \frac{1}{4}\left[\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix} - \begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}\right]$$

$$= \frac{1}{4}\left[\begin{pmatrix}0&0&0\\0&i&0\\0&0&-i\end{pmatrix} - \begin{pmatrix}0&0&0\\0&-i&0\\0&0&i\end{pmatrix}\right] = \frac{i}{2}\begin{pmatrix}0&0&0\\0&1&0\\0&0&-1\end{pmatrix}$$

$$= \frac{i}{2}\left[f\underbrace{\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}}_{\lambda_3} + f'\underbrace{\frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}}_{\lambda_8}\right]$$

$$= \text{RHS for } f = \boxed{f_{673} = \frac{-1}{2}}$$

$$f' = \boxed{f_{678} = \frac{\sqrt{3}}{2}}$$

$i=1,\ j=4$

$$\left[\frac{\lambda_1}{2}, \frac{\lambda_4}{2}\right] = \frac{1}{4}\left[\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix} - \begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}\right]$$

$$= \frac{1}{4}\left[\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix} - \begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix}\right] = -\frac{i}{4}\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix}$$

$$= \frac{i}{2}\left[f\underbrace{\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix}}_{\lambda_7}\right] = \text{RHS for}$$

$$f = \boxed{f_{147} = \frac{1}{2}}$$

$i = 1, \; j = 6$

$$\left[\frac{\lambda_1}{2}, \frac{\lambda_6}{2}\right] = \frac{1}{4}\left[\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right]$$

$$= \frac{1}{4}\left[\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\right] = -\frac{i}{4}\underbrace{\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}}_{\lambda_5}$$

$$= i f \lambda_5 \quad = \text{RHS for } f = \boxed{f_{165} = \frac{1}{2}}$$

$i = 2, \; j = 4$

$$\left[\frac{\lambda_2}{2}, \frac{\lambda_4}{2}\right] = \frac{1}{4}\left[\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right]$$

$$= \frac{1}{4}\left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -i & 0 \end{pmatrix}\right] = \frac{i}{4}\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}_{\lambda_6}$$

$$= i f \lambda_6 \quad = \text{RHS for } f = \boxed{f_{246} = \frac{1}{2}}$$

$i = 2, \; j = 5$

$$\left[\frac{\lambda_2}{2}, \frac{\lambda_5}{2}\right] = \frac{1}{4}\left[\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right]$$

$$= \frac{1}{4}\left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}\right] = \frac{i}{4}\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}}_{\lambda_7}$$

$$= i f \lambda_7 \quad = \text{RHS for } f = \boxed{f_{257} = \frac{1}{2}}$$

Problem 5   Halzen & Martin 14.13

ϕ : SU(2) triplet of real scalar fields

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$$

Use the Higgs potential $V(\phi) = \mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2$,

$\mu^2 < 0$, $\lambda > 0$

3-D rep of SU(2) is $(T_k)_{ij} = -i\varepsilon_{ijk}$, (satisfies SU(2) algebra)

or written out as 3 matrices

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The analog of the covariant derivative is

$\mathcal{D}_\mu = \partial_\mu + ig\,\vec{T}\cdot\vec{W}_\mu$, and it appears as $\frac{1}{2}(\mathcal{D}_\mu\phi)^\dagger(\mathcal{D}^\mu\phi) - V(\phi)$ in the Lagrangian

When the symmetry is broken by choosing a ground state $\phi_0 = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}$,

the term which will generate masses is then

$$\frac{1}{2}\left| ig\,\vec{T}\cdot\vec{W}_\mu \phi \right|^2 = \frac{1}{2} g^2 \left| \left(T_1 W_{1\mu} + T_2 W_{2\mu} + T_3 W_{3\mu}\right) \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} \right|^2$$

convention

$$= \frac{1}{2} g^2 \left| \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -iW_{1\mu} \\ 0 & iW_{1\mu} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & iW_{2\mu} \\ 0 & 0 & 0 \\ -iW_{2\mu} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -iW_{3\mu} & 0 \\ iW_{3\mu} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} \right|^2$$

$$= \frac{1}{2} g^2 \left| \begin{pmatrix} 0 & -iW_{3\mu} & iW_{2\mu} \\ iW_{3\mu} & 0 & -iW_{1\mu} \\ -iW_{2\mu} & iW_{1\mu} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} \right|^2$$

$$= \frac{1}{2} g^2 v^2 \left(-iW_{2\mu}, iW_{1\mu}, 0 \right) \begin{pmatrix} iW_{2\mu} \\ -iW_{1\mu} \\ 0 \end{pmatrix}$$

$$= \frac{1}{2} g^2 v^2 \left(W_{2\mu}{}^2 + W_{1\mu}{}^2 \right) \longrightarrow \text{of the form } \tfrac{1}{2} M^2 W^2$$

This Lagrangian term corresponds to two massive gauge bosons with mass $\boxed{M_1 = M_2 = gv}$

The third gauge boson has no mass term and so remains massless $\boxed{M_3 = 0}$