Problem Set #4 Solutions

Problem 1 Halzen & Martin 15.2

To show: in the Weinberg-Salam model

$$\frac{1}{2v^2} = \frac{g^2}{8M_W^2} = \frac{G}{\sqrt{2}}$$

and verify that $v = 246$ GeV

Derive $M_W = \dfrac{37.3}{\sin\theta_W}$ GeV, $M_Z = \dfrac{74.6}{\sin 2\theta_W}$ GeV

and give lower bounds for their masses; predict M_W & M_Z using experimental $\sin^2\theta_W$

H&M (12.15) $$\frac{G}{\sqrt{2}} = \frac{g^2}{8M_W^2}$$

15.18 $$M_W = \frac{1}{2} v g$$

$$\frac{G}{\sqrt{2}} = \frac{g^2}{8\left(\frac{1}{4}v^2 g^2\right)}$$

$$\Rightarrow \boxed{\frac{1}{2v^2} = \frac{g^2}{8M_W^2} = \frac{G}{\sqrt{2}}}$$ as required

Empirical value of G: H&M $G = 1.166 \times 10^{-5}\ \text{GeV}^{-2}$

$$\Rightarrow v = \left(\frac{\sqrt{2}}{2G}\right)^{1/2} = \left(\frac{\sqrt{2}}{2\,(1.166\times 10^{-5})}\right)^{1/2} = \underline{\underline{246\ \text{GeV}}}$$

$$M_W = \frac{1}{2} v g \,, \qquad 15.9 \qquad g = \frac{e}{\sin\theta_W}$$

$$\Rightarrow M_W = \frac{1}{2} v \frac{e}{\sin\theta_W}$$

$$\alpha = \frac{e^2}{4\pi} = \frac{1}{137}$$

$$= \frac{1}{2} \frac{(246)\sqrt{4\pi\alpha}\ \text{GeV}}{\sin\theta_W}$$

$$\boxed{M_W = \frac{37.3}{\sin\theta_W}\ \text{GeV}}$$

$$M_Z = \frac{M_W}{\cos\theta_Z} = \frac{37.3\ \text{GeV}}{\sin\theta_W \cos\theta_W} \qquad 2\sin\theta_W\cos\theta_W = \sin 2\theta_W$$

$$\boxed{M_Z = \frac{74.6}{\sin 2\theta_W}\ \text{GeV}}$$

Lower bounds for masses: $\sin\theta_W = 1$ gives lowest possible $M_W = 37.3$ GeV

$\sin 2\theta_W = 1$ gives lowest possible $M_Z = 74.6$ GeV (independently of M_W min)

For $\sin^2\theta_W = 0.231$ (latest PDB value)

$\sin\theta_W = 0.4806$

$\cos\theta_W = \sqrt{1 - 0.231}$

$$\boxed{M_W = 77.6\ \text{GeV}}$$

$$M_Z = \frac{M_W}{\cos\theta_W} \Rightarrow \boxed{M_Z = 88.5\ \text{GeV}}$$

Problem 2 Show that Yukawa Lagrangian

$$\mathcal{L} = -G_e \left[(\bar{\nu}_e, \bar{e})_L \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} e_R + \bar{e}_R (\phi^-, \bar{\phi}^0) \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \right]$$

is $SU(2)_L \times U(1)_Y$ invariant

$SU(2)_L \times U(1)_Y$ transformation: $\chi_L \to \chi_L' = e^{i\vec{\alpha}\cdot\vec{T} + i\beta Y} \chi_L$

$\psi_R \to \psi_R' = e^{i\beta Y} \psi_R$

Higgs doublet $\begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$ has $Y_2 = 1$, $T = 1/2$

$Y = 2(Q - T_3)$

ϕ^+ : $Q = 1$, $T_3 = 1/2$, $Y = 2(1 - 1/2) = 1$

ϕ^0 : $Q = 0$, $T_3 = -1/2$, $Y = 2(0 + 1/2) = 1$

Lepton doublet $\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$ has $Y_1 = -1$, $T = 1/2$

e_R has $Y_3 = -2$, $T = 0$

$$\mathcal{L} \xrightarrow{SU(2)\times U(1)} \mathcal{L}' = -G_e \Big[(\bar{\nu}_e \ \bar{e})_L e^{-i(\vec{\alpha}\cdot\vec{T} + \beta Y_1)} e^{i(\vec{\alpha}\cdot\vec{T} + i\beta Y_2)} \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} e^{i\beta Y_3} e_R$$

$$+ \bar{e}_R e^{-i\beta Y_3} (\phi^- \ \bar{\phi}^0) e^{-i(\vec{\alpha}\cdot\vec{T} + i\beta Y_2)} e^{i(\vec{\alpha}\cdot\vec{T} + i\beta Y_1)} \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \Big]$$

$$= -G_e \Big[e^{i\beta(-Y_1 + Y_2 + Y_3)} (\bar{\nu}_e \ \bar{e})_L \underbrace{e^{-i(\vec{\alpha}\cdot\vec{T})} e^{i(\vec{\alpha}\cdot\vec{T})}}_{1} \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} e_R$$

$$+ e^{i\beta(-Y_3 - Y_2 + Y_1)} \bar{e}_R (\phi^-, \bar{\phi}^0) \underbrace{e^{-i(\vec{\alpha}\cdot\vec{T})} e^{i(\vec{\alpha}\cdot\vec{T})}}_{1} \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \Big]$$

$$-Y_1 + Y_2 + Y_3 = 1 + 1 - 2 = 0$$

$$\Rightarrow e^{i\beta(-Y_1 + Y_2 + Y_3)} = 1$$

Similarly, $e^{i\beta(-Y_3 - Y_2 + Y_1)} = 1$

So $\mathcal{L}' = -G_e\left[(\bar{\nu}_e \ \bar{e})_L \begin{pmatrix}\phi^+ \\ \phi^0\end{pmatrix} e_R + \bar{e}_R(\phi^- \ \bar{\phi}^0)\begin{pmatrix}\nu_e \\ e\end{pmatrix}_L\right]$

$$\mathcal{L}' = \mathcal{L}$$

$\Rightarrow$ $\boxed{\mathcal{L}_{\text{Yukawa}} \text{ is invariant under } SU(2)_L \times U(1)_Y}$

Problem 3 Halzen & Martin 15.6

Derive (15.39) $M_h^2 = 2v^2\lambda$, see (14.58)

$$\mathcal{L}'' = \tfrac{1}{2}(\partial_\mu h)^2 - \lambda v^2 h^2 + \tfrac{1}{2}e^2 v^2 A_\mu^2 - \lambda v h^3$$
$$- \tfrac{1}{4}\lambda h^4 + \tfrac{1}{2}e^2 A_\mu^2 h^2 + v e^2 A_\mu h - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

(Here using Halzen & Martin notation)

Assuming, $V(\phi) = \mu^2 \phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2 + \dots$

The 1D Higgs mechanism analysis of section 14.8 applies in exactly the same way.

The Higgs mass term in (14.58) is $\frac{1}{2}m_h^2 = \lambda v^2$

$$\Rightarrow \boxed{m_h = 2v^2\lambda}$$

Problem 4

$p \to e^+ \pi^0$ decay at rest : KE of π^0 and e^+ is $m_p - m_{e^+} - m_{\pi^0} \sim 800$ MeV, but in fact e^+ annihilates and $\pi^0 \to \gamma\gamma$, and γ energy will be absorbed.

$\Rightarrow$ Assume that all of the proton rest mass energy will be deposited in your body

If 5 rad/yr is bad, then say ~100 rad in 4 months will do you in.

$$\frac{100 \text{ rad}}{1/3 \text{ yr}} \times 100 \frac{\text{ergs/g}}{\text{rad}} \times 10^{-7} \frac{\text{J}}{\text{erg}} \times \frac{1 \text{ eV}}{1.6 \times 10^{-19} \text{J}} \times \frac{1 \text{ pdk}}{938 \times 10^6 \text{eV}}$$

$\sim 2 \times 10^7$ pdk/(yr·g) will kill you

There are 6×10^{23} protons / g

Most nuclei are ~1/2 protons, so your body has $\sim 3 \times 10^{23}$ protons / g

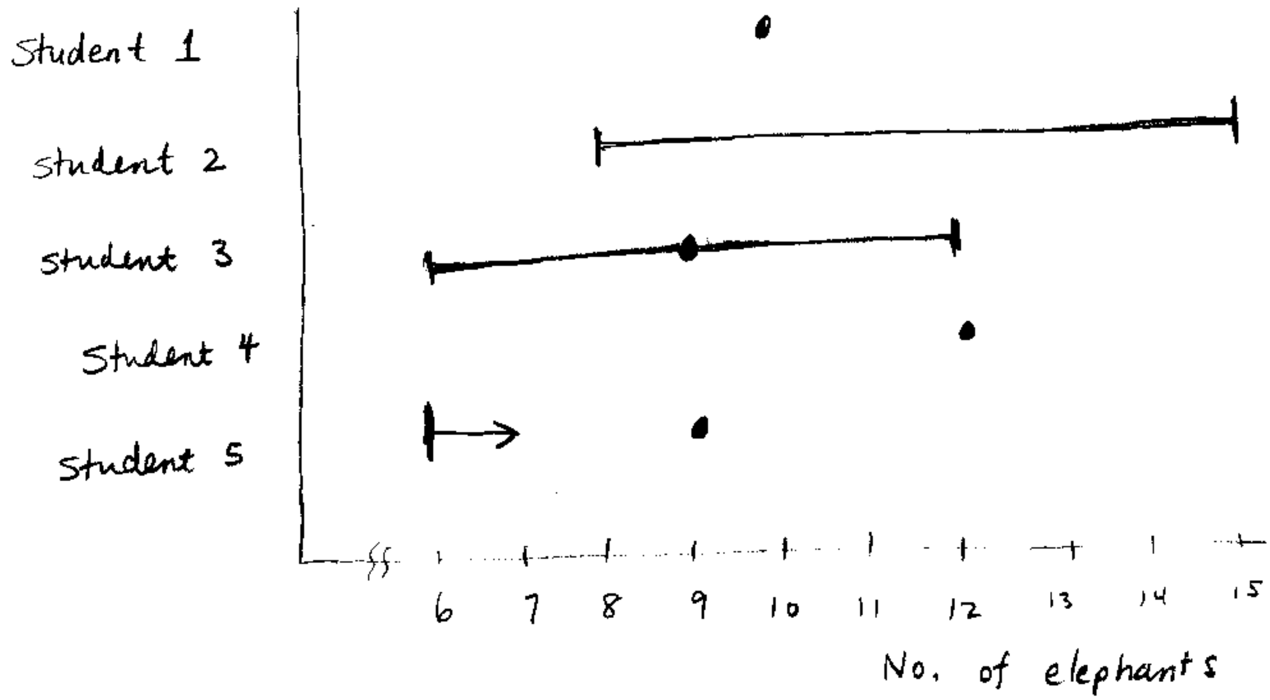
$$2 \times 10^7 \frac{\text{pdk}}{\text{yr} \cdot \text{g}} \times \frac{1 \text{ g of body matter}}{3 \times 10^{23} \text{ protons}} \sim 10^{-16} \frac{\text{pdk}}{\text{yr}}$$

$\Rightarrow$ Allowing a factor of a few for lifetime limit,

$\boxed{\tau_p \gtrsim 10^{16} \text{ yr}}$ or else 8.8 would certainly kill you

(other answers OK within few orders of magnitude for reasonable assumptions)

Problem 5



Results are consistent.