Problem Set #5 Solutions

$p \to e^+ \pi^0$ Assume proton decays at rest.

First we need to calculate the energy of the decay products. In a 2-body decay, the π^0 and e^+ get equal momentum. $\pi^0 \to \gamma\gamma$ with very short lifetime.

$$|\vec{p}_\pi| = |\vec{p}_{e^+}| = |\vec{p}|$$

Mass-energy available is $m_p = \sqrt{|\vec{p}_\pi|^2 + m_\pi^2} + \sqrt{|\vec{p}_e|^2 + m_e^2}$

$$m_p^2 + \cancel{p^2} + m_e^2 - 2m_p\sqrt{p^2 + m_e^2} = \cancel{p^2} + m_\pi^2$$

$$2m_p\sqrt{p^2+m_e^2} = m_p^2 - m_\pi^2 + m_e^2$$

$$\sqrt{p^2+m_e^2} = \frac{1}{2}\left(m_p - \frac{m_\pi^2}{m_p} + \frac{m_e^2}{m_p}\right)$$

$$p = \left[\frac{1}{4}\left(m_p - \frac{m_\pi^2}{m_p} + \frac{m_e^2}{m_p}\right)^2 - m_e^2\right]^{1/2}$$

Plugging in $m_p = 938.3\ \frac{\text{MeV}}{c^2}$, $m_\pi = 135\ \frac{\text{MeV}}{c^2}$, $m_e = 0.511\ \frac{\text{MeV}}{c^2}$

$$|\vec{p}| = |\vec{p}_\pi| = |\vec{p}_{e^+}| = 459\ \text{MeV}/c$$

At this energy, you'll get em showers (E_c in water ~ 92 MeV) for both the positrons and the π^0 dk γ's.

The 'peak' of a shower occurs at

$$t_{max} \sim \frac{\ln \frac{E_0}{E_c}}{\ln 2},$$

(in radiation lengths)

where E_c is the critical energy at E_0 is the incident energy

Take the positron $E_0 \sim 460$ MeV (this energy is divided for the $\pi^0 \to \gamma\gamma$ gammas)

$$t_{max} = \frac{\ln \frac{460}{92}}{\ln 2} = 2.3\ X_0$$

X_0 in water: see PDG 2002 p. 199, eqn (26.20)

$$X_0 = \frac{716\ g\ cm^{-2}\ A}{Z(Z+1)\ln(287/\sqrt{Z})}$$

For a compound, $1/X_0 = \Sigma^{w_j}/X_j$, where w_j is the fraction of each element by weight

For water, $X_0 \sim 36\ cm^{-2}\ g$; $\rho_{water} = 1\ g/cm^2$

$\Rightarrow t_{max} \sim 83$ cm

There will be back-to-back showers, so to contain the showers detector will need to be at least a few meters wide, and more if you want some sort of reasonable fiducial 'target' volume. Let's say ~5-10m on a side.

Many people estimated the Cherenkov photons as follows: total track length of showers: ~ 2 m

Cherenkov emission yields roughly 200 photons / cm
(Perkins p.56)

So expect (200)(200) ~ 40000 photons

However this isn't exactly correct, since individual particle tracks in the shower add up to more — still it's a lower limit (and actually gives not too crazy an answer)

Another way of estimating: expect 938 MeV of energy loss, total, from one pdk. Energy loss to Ch. light is going to be only a small fraction of this total ⇒ Perkins p.56, fraction is

$$\frac{\text{Ch.}}{\text{ioniz.}} \sim \frac{400\ \text{eV/cm}^{-1}}{2\ \text{MeV/cm}^{-1}} \sim 2\times10^{-4}$$

So expect $\sim(2\times10^{-4})\,938 \sim 0.2$ MeV to go into Cherenkov light.

Photons of 400-700 nm have about 2 eV apiece.

So 0.2 MeV represent $\sim 10^5$ photons

(off by a factor of 2 from above.)

We are told that optical transmission is 20%. (assume for relevant length scale, which is not in the problem statement) and Q.E. is 15%.

So if the surface were completely covered by photocathode, we'd get $(10^5)(0.2)(0.15) = 3000$ photons.

Assume energy resolution is dominated by p.e. statistics; for $\frac{N_{ph}}{\sqrt{N_{ph}}} \lesssim 10\%$ resolution,

need $\gtrsim 100$ photons per event.

So the minimum required photocathode coverage is $\frac{100}{3000} \sim \underline{\underline{3\%}}$ (answers from a few to ~15% OK for reasonable assumptions)

Note: Super-K I (before the accident) had ~40% photocathode coverage, required for sufficient resolution at ~MeV energies for solar ν's